Problem Sheet 11

Problem 1

Let p be an odd prime.

(a) Given $n \geq 1$, take its p-adic expansion $n = a_0 + a_1 p + \ldots + a_r p^r$, $0 \leq a_i \leq p-1$, and define $s_p(n) = \sum_{i=0}^r a_i$. Show that $v_p(n!) = (n - s_p(n))/(p-1)$.

(b) Prove that for $x \in \mathbb{Z}_p$, the series

$$(1+p)^x := 1 + \sum_{n=1}^{\infty} \frac{x(x-1)\cdots(x-n+1)}{n!} p^n$$

converges to an element in $\mathbb{Z}_p$ and that $(1+p)^x(1+p)^y = (1+p)^{x+y}$.

(c) Deduce that $x \mapsto (1+p)^x$ defines an isomorphism of abelian groups $(\mathbb{Z}_p, +) \cong (1+p\mathbb{Z}_p, \times)$ and conclude that

$$\mathbb{Z}_p^\times \cong (\mathbb{Z}/p)^\times \times \mathbb{Z}_p.$$

Problem 2

Let $K = \cup_n \mathbb{C}((t^{1/n}))$ be the field of Puiseux series; set $\mathcal{O}_K := \cup_n \mathbb{C}[[t^{1/n}]]$.

(a) Show that $v\colon K \to \mathbb{Q} \cup \{\infty\}$, $v(f) := \deg(f)$ defines a valuation with valuation ring $\mathcal{O}_K$. Prove that the maximal ideal of $\mathcal{O}_K$ is not finitely generated.

(b) Let $\widehat{\mathcal{O}_K}$ be the t-adic completion of $\mathcal{O}_K$. Write down an element of $\widehat{\mathcal{O}_K} \setminus \mathcal{O}_K$.

Problem 3

(a) Let R be a ring and $f \in R$ a non-zero divisor. Assume that R is (f)-adically complete and that $\overline{R} = R/(f)$ is a perfect ring in characteristic p. (This means that the Frobenius $x \mapsto x^p$ defines an automorphism of $\overline{R}$.) Show that there is a unique multiplicative section $[\cdot]\colon \overline{R} \to R$ to the projection $R \to \overline{R}$. It is called the Teichmüller lift.
Hint: For $r \in \overline{R}$, denote by $\tilde{r} \in R$ any lift. Show that the following sequence converges,

$$\left(\widetilde{r^{1/p^n}}\right)^{p^n}.$$

(b) Assume R is itself of characteristic p. Show that the Teichmüller lift is then a ring homomorphism and that $R \cong \overline{R}[[f]]$.

(c) Assume that $F/\mathbb{Q}_p$ is a finite extension and that $R = \mathcal{O}_F$ with uniformizer π and residue field $\mathbb{F}_q$. Show that for every $x \in \mathbb{F}_q$ there is a unique $(q-1)$-st root of unity $\zeta \in R$ with $\zeta \equiv x \mod \pi R$ and that $[x] = \zeta$.